A Cantor set in the unit sphere in $\mathbb{C}^2$ with large polynomial hull *†

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Abstract

It is an old question how massive polynomial hulls of Cantor sets in $\mathbb{C}^n$ can be. In contrast to expectation e.g. Rudin, Vitushkin and Henkin showed on examples that it can be rather massive. Motivated by problems of holomorphic convexity of subsets of strictly pseudoconvex boundaries and removable singularities the question was asked for Cantor sets in the unit sphere. It was known that tame Cantor sets in the unit sphere are polynomially convex. We give an example of a wild Cantor set in the sphere whose polynomial hull contains a large ball. In some sense this can be opposed to a still open conjecture of Vitushkin on the existence of a lower bound for the diameter of the largest boundary component of a relatively closed complex curve in the ball passing through the origin.

It is an old question by Walter Rudin asked in connection with Banach algebras and approximation by polynomials how massive polynomial hulls of Cantor sets may be. In contrast to expectation (note that Cantor sets have Hausdorff dimension zero) e.g. Rudin, Vitushkin and Henkin showed on examples that the mentioned polynomial hull can be rather massive. Rudin himself constructed a Cantor set in $\mathbb{C}^2$ such that its polynomial hull (and even its rational hull) contains an analytic variety of dimension one ([12], theorem 5; [7], theorem III.2.5). Later Vitushkin [14] and Henkin [8] gave examples of Cantor sets with interior points in the polynomial hull.

The problem received new attention in connection with interest in topology on strictly pseudoconvex boundaries and hulls of their subsets as well as in connection with removable singularities of CR functions. In particular, it was asked whether Cantor sets in the unit sphere $\mathbb{C}^2$ are polynomially convex. The expectation was that for subsets of the sphere the situation changes dramatically as it is the case for some other problems. E.g. totally real discs in $\mathbb{C}^2$ are not necessarily polynomially convex [11], but if contained in the sphere they are so [9]. Further, the polynomial hull of a compact set in $\mathbb{C}^2$ of finite one-dimensional Hausdorff measure is not necessarily an analytic variety [1], but if the set is contained in the sphere this is so [1]. Moreover, the question about polynomial hulls of Cantor sets in the sphere has some relation to a still open conjecture of Vitushkin on the existence of

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a lower bound for the diameter of the largest boundary component of a relatively closed complex curve in the ball passing through the origin.

In [6] the slightly more general question was raised whether Cantor sets in boundaries ∂G of strictly pseudoconvex domains G in $\mathbb{C}^2$ are convex with respect to the space of holomorphic functions in G which are continuous in the closure $\overline{G}$ of the domain G. The question was answered in [6] affirmatively for some class of Cantor sets. The main tools used for this in [6] are the theorem of Bedford and Klingenberg [2], a characterization of tame Cantor sets in $\mathbb{R}^n$ due to Bing and the non-trivial fact that a C^2 manifold that is homeomorphic to $\mathbb{R}^3$ is C^2 diffeomorphic to $\mathbb{R}^3$ (see references in [6]). With the mentioned tools in mind it was a natural step to obtain the following result.

Theorem a. *Tame Cantor sets in the unit sphere in $\mathbb{C}^2$ are polynomially convex.*

The theorem follows directly from [5] and the mentioned result of Bing and was proved independently a bit later by M. Lawrence [10].

Recall that a compact set $E \subset \mathbb{R}^n$ is a tame Cantor set if there is a homeomorphism of $\mathbb{R}^n$ which carries E to the middle-third Cantor set in a coordinate line. Bing's result is that this is equivalent to the following separation property.

For each pair of different points p and q in E and any $\varepsilon > 0$ there exists a set $\overline{b^n} \subset \mathbb{R}^n$ homeomorphic to a closed n-ball of diameter less than ε with boundary $\partial b^n = \mathcal{S}^{n-1}$ disjoint from E with $p \in b^n$ and $q \notin \overline{b^n}$.

Theorem a can be stated in a slightly more general form. For a domain $G \subset \mathbb{C}^n$ denote by $A(G)$ the algebra of analytic functions in G which are continuous in $\overline{G}$. For a compact subset K of $\overline{G}$ we consider its $A(G)$-hull,

$$A(G) - \mathrm{hull}(K) = \{z \in \overline{G} : |f(z)| \leq \max_K |f| \text{ for all functions } f \in A(G)\}.$$

In case $G = \mathbb{C}^n$ we obtain the polynomial hull

$$\hat{K} = \{z \in \mathbb{C}^n : |p(z)| \leq \max_K |p| \text{ for all polynomials } p\}$$

of the compact subset K of $\mathbb{C}^n$.

The rational hull of the compact K is defined by replacing in the last condition polynomials by rational functions which are analytic in a neighbourhood of the compact K.

Theorem b. *Let G be a bounded strictly pseudoconvex domain in $\mathbb{C}^2$ with smooth boundary. Let $K \subset \partial G$ be a Cantor set with the following separation property.*

For each pair of distinct points p and q in K there exists a smooth 2-sphere $\mathcal{S}^2 \subset \partial G \backslash K$ which separates p and q (i.e. p and q are in different connected components of $\partial G \setminus \mathcal{S}^2$).

Then K is $A(G)$-convex (i.e. $A(G) - hull(K) = K$).

Theorem a follows from theorem b by the result of Bing and the mentioned fact about homeomorphic and diffeomorphic $\mathbb{R}^3$ (for more detail see [6]). Note that it is not necessary to require in theorem b that the 2-spheres bound 3-balls in ∂G and that these balls have small size. In the description of tame Cantor sets the condition that the b^3 has small

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diameter is important. There are wild Cantor sets in $\mathbb{R}^3$ ([3]) each pair of distinct points of which can be separated by spheres (but not by spheres of small size).

The proof of theorem b follows from the theorem of Bedford and Klingenberg. Here is a sketch of its proof for convenience of the reader (see [6], [5], [10]).

If K were not $A(G)$-convex, then by Zorn's lemma there would be a minimal compact subset K' of K containing a given point of G in its $A(G)$-hull. Then $A(G)$-hull(K') is connected by Rossi's maximum principle. By the separation property there is a smooth 2-sphere $\mathcal{S}^2 \subset \partial G \setminus K'$ which divides K' and hence also ∂G. By [2] (maybe after perturbing the 2-sphere) $\mathcal{S}^2$ bounds a Levi-flat 3-ball $\mathcal{B}^3 \subset G$ such that $\overline{\mathcal{B}}^3$ is equal to the envelope of holomorphy of $\mathcal{S}^2$ and to $A(G)$-hull($\mathcal{S}^2$). By a theorem of Alexander and Stout (see the references in [13]) $\overline{\mathcal{B}}^3$ divides $\overline{G}$. The envelope of holomorphy of $\partial G \setminus K'$ equals $\overline{G} \setminus A(G) - \text{hull}(K')$ (see the references in [13]), hence $\overline{\mathcal{B}}^3$ does not meet $A(G)$-hull(K'). Hence $A(G)$-hull(K') is contained in a connected component of $\overline{G} \setminus \overline{\mathcal{B}}^3$ which is impossible.

The mentioned results suggested that e.g. Cantor sets in the unit sphere in $\mathbb{C}^2$ are polynomially convex. The purpose of this paper is to construct Cantor sets in the unit sphere with large polynomial hull.

For a subset A of $\mathbb{C}^n$ and a positive number r we denote by rA the set $\{rz : z \in A\}$. Let $\mathbb{B}^n$ denote the unit ball in $\mathbb{C}^n$ and $\partial\mathbb{B}^n$ its boundary.

Theorem. *For any positive number $\beta < 1$ there exists a Cantor set E contained in $\partial\mathbb{B}^2$ whose polynomial hull $\hat{E}$ contains the closed ball $\overline{\beta\mathbb{B}^2}$.*

The theorem implies e.g. that there are continuous arcs in the sphere (i.e. homeomorphic images of the unit interval of the real line) whose polynomial hull contains big balls.

The basis of the proof is the following observation. Consider two circles on the sphere, each of them being the intersection of the sphere with a complex line. Then the polynomial hull of their union is equal to the union of their polynomial hulls. However, the polynomial hull of the union of their ε-neighbourhoods ($\varepsilon > 0$) is essentially larger than the union of the hull of their ε-neighbourhoods. More precisely, the following lemma holds.

Main lemma. *Let f and g be complex affine functions on $\mathbb{C}^2$ with $|\nabla f| = |\nabla g| = 1$. Suppose the sets $\{f = 0\} \cap \partial\mathbb{B}^2$ and $\{g = 0\} \cap \partial\mathbb{B}^2$ are disjoint circles. Then there exist positive numbers $a = a(f, g)$ and $r' = r'(f, g) < 1$ such that for any positive ε the following inclusion holds:*

$$(1) \qquad \{|f \cdot g| \leq a\varepsilon\} \cap (\overline{\mathbb{B}^2} \setminus r'\mathbb{B}^2) \subset (\{|f| \leq \varepsilon\} \cup \{|g| \leq \varepsilon\}) \cap (\overline{\mathbb{B}^2} \setminus r'\mathbb{B}^2).$$

By a complex affine function f we mean a mapping $f : \mathbb{C}^2 \to \mathbb{C}$ of the form $f(z) = f_0 + f_1 \cdot z_1 + f_2 \cdot z_2$ for complex numbers f_0, f_1 and f_2.

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Corollary 1. *With f, g, a, ε and r' as above the inclusion*

$$(2) \qquad \{|fg| \le a\varepsilon\} \cap \overline{r\mathbb{B}^2} \subset \left((\{|f| \le \varepsilon\} \cap \widehat{r\partial\mathbb{B}^2}) \cup (\{|g| \le \varepsilon\} \cap r\partial\mathbb{B}^2) \right)$$

holds for any $r \in [r', 1]$.

Proof. (1) implies in particular that for any $r \in [r', 1]$

$$(3) \qquad \{|fg| \le a\varepsilon\} \cap r\partial\mathbb{B}^2 \subset (\{|f| \le \varepsilon\} \cap r\partial\mathbb{B}^2) \cup (\{|g| \le \varepsilon\} \cap r\partial\mathbb{B}^2)$$

holds. The polynomial hull of the left hand side is $\{|fg| \le a\varepsilon\} \cap \overline{r\mathbb{B}^2}$, hence (2) holds. $\square$.

Remark 1. *Each set on the right hand side of (3) is the intersection of the sphere $r\partial\mathbb{B}^2$ with the closed ε-neighbourhood of a circle (i.e. a solid torus in the 3-sphere if ε is small). Suppose the complex lines $\{f = 0\}$ and $\{g = 0\}$ intersect inside $\mathbb{B}^2$. Then the left hand side of (2) contains e.g. a ball of radius $\sqrt{a} \cdot \sqrt{\varepsilon}$ around the intersection point. The radius is much larger than ε for small enough $\varepsilon > 0$. The proof of the theorem is based on this observation.*

Proof of the main lemma. Since the circles are disjoint there exist a and r' such that also the sets $\{|f| \le a\} \cap (\overline{\mathbb{B}^2} \setminus r'\mathbb{B}^2)$ and $\{|g| \le a\} \cap (\overline{\mathbb{B}^2} \setminus r'\mathbb{B}^2)$ are disjoint. If $z \in \overline{\mathbb{B}^2} \setminus r'\mathbb{B}^2$ and $|f(z) \cdot g(z)| \le a \cdot \varepsilon$, then either $|f(z)| > a$ and then $|g(z)| \le \varepsilon$, or $|f(z)| \le a$ and then $|g(z)| > a$ and hence $|f(z)| \le \varepsilon$. $\square$.

Note that the constants a and r' can be chosen so that they serve also for pairs of complex affine functions close to f and g.

Preparation of the proof of the theorem. We will find the required set E of the form $E = \bigcap\limits_{N=1}^{\infty} E_N$ for a decreasing family of closed sets $E_N \subset \partial\mathbb{B}^2$ such that $\overline{\beta\mathbb{B}^2} \subset \hat{E}_N$ for each N. Then $\hat{E} \supset \overline{\beta\mathbb{B}^2}$. Indeed, let $z \in \overline{\beta\mathbb{B}^2}$. For any fixed polynomial p and for any N there exists a point $z_N \in E_N$ for which $|p(z)| \le |p(z_N)|$. If (for fixed p) z^* is an accumulation point of the z_N, then $z^* \in E$ and by continuity $|p(z)| \le |p(z^*)|$. This holds for arbitrary polynomials p, hence $z \in \hat{E}$.

Each set E_N will be the finite union of disjoint solid tori of the form described above. Introduce the following notation. For a complex affine function f with $|\nabla f| = 1$ and a positive number σ we denote $T_f(\sigma) = \{z \in \partial\mathbb{B}^2 : |f(z)| \le \sigma\}$. After a unitary change of coordinates in $\mathbb{C}^2$ we may assume that $|f|$ has the form $|z_1 - (1 - s)|$ for a real number $s = s_f$. The mentioned unitary transformation takes $T_f(\sigma)$ to the set $T^s(\sigma) = \{z \in \partial\mathbb{B}^2 : |z_1 - (1 - s)| \le \sigma\}$. This is a solid torus if $s < 1$ and $\sigma < s$. For $r < 1$ denote $^rT_f(\sigma) = \{|f| \le \sigma\} \cap r\partial\mathbb{B}^2$. Similarly we write $^rT^s(\sigma)$. Denote finally the complex lines (the symmetry axes of the respective tori) by $\ell_f = \{f = 0\}$ and $\ell^s = \{z_1 = 1 - s\}$.

Note that the number σ together with the unitary invariant s of a torus $T_f(\sigma)$ contained in $\partial\mathbb{B}^2$ determine its diameter. If s and σ are small then the diameter of $T^s(\sigma)$ (hence of each unitarily equivalent torus) is small.

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By the following easy lemma it is enough to cover bidiscs by polynomial hulls of suitable Cantor sets.

Lemma 1. *For any $\beta \in (0,1)$ there is a number $q \in (0,1)$ such that $\overline{\beta \mathbb{B}^2}$ can be covered by a finite union of bidiscs of the form $q(\overline{\mathcal{D}_1} \times \overline{\mathcal{D}_2})$ where $\overline{\mathcal{D}}_j$ are closed discs in $\mathbb{C}$ centered at zero such that $\partial \mathcal{D}_1 \times \partial \mathcal{D}_2 \subset \partial \mathbb{B}^2$.* $\qquad\square$

The following proposition allows to cover bidiscs of lemma 1 by the polynomial hull of finite unions of disjoint solid tori which are arbitrarily thin tubular neighbourhoods of (in general not small) circles.

Proposition 1. *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be discs in $\mathbb{C}$ centered at the origin and such that $\partial \mathcal{D}_1 \times \partial \mathcal{D}_2 \subset \partial \mathbb{B}^2$. Let $\gamma > 0$ and $q \in (0,1)$. There exist two numbers $s_1, s_2 \in (0,1)$ and a positive number ε', all depending only on $\mathcal{D}_1, \mathcal{D}_2, q$ and γ, such that for each $\varepsilon \in (0, \varepsilon')$ there exist two families of solid tori $T_j(\varepsilon) = T_{f_j}(\varepsilon)$ and $T_k^*(\varepsilon) = T_{g_k}(\varepsilon)$, each family containing finitely many tori which are unitarily equivalent to $T^{s_1}(\varepsilon)$ and $T^{s_2}(\varepsilon)$, respectively, with the following properties:*

- *All thicker tori $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are pairwise disjoint and contained in the γ-neighbourhood of $\partial \mathcal{D}_1 \times \partial \mathcal{D}_2$.*

- *The polynomial hull of the union satisfies the relation*

$$
(4) \qquad q(\overline{\mathcal{D}_1} \times \overline{\mathcal{D}_2}) \subset \widehat{\left(\bigcup_j T_j(\varepsilon) \cup \bigcup_k T_k^*(\varepsilon) \right)}.
$$

Remark 2. *For some $r' < 1$ depending only on $\mathcal{D}_1, \mathcal{D}_2, q$ and γ the tori in proposition 1 can be chosen such that for all $r \in [r', 1]$ also the inclusion*

$$
(4_r) \qquad q(\overline{\mathcal{D}_1} \times \overline{\mathcal{D}_2}) \subset \widehat{\left(\bigcup_j {}^r T_j(\varepsilon) \cup \bigcup_k {}^r T_k^*(\varepsilon) \right)}
$$

holds.

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and $\mathbb{T} = \partial \mathbb{D}$.

Proof of proposition 1. Let $\mathcal{D}_1 = R_1 \mathbb{D}$, $\mathcal{D}_2 = R_2 \mathbb{D}$. Increasing q we may assume that $q < 1$ is as close to 1 as needed. Let ζ_1^j be equidistributed points on $q R_1 \mathbb{T}$ with distance of nearest points being a number between $B\varepsilon$ and $(B+1)\varepsilon$, where B is any constant, $B \geq 5$, and $\varepsilon > 0$ is small enough. If ε is small for each B such points can be found. Similarly, let ζ_2^k be equidistributed points on $q R_2 \mathbb{T}$ with distance between closest points in $[B\varepsilon, (B+1)\varepsilon]$. Define

$$
f_j(z) = z_1 - \zeta_1^j, \quad g_k(z) = z_2 - \zeta_2^k.
$$

All sets $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are non-empty tori if ε is small enough and they are pairwise disjoint for such ε. Disjointness is clear for tori of the same family since $B \geq 5$ and follows from the fact that $\{f_j = 0\} \cap \{g_k = 0\}$ is contained in $\mathbb{B}^2$ in the other case.

For any j and k there is a unitary transformation which takes $|f_j|$ to $|z_1 - (1 - s_1)|$, $s_1 = 1 - R_1 q$, and $|g_k|$ to $|z_2 - (1 - s_2)|$, $s_2 = 1 - R_2 q$. It follows that the $T_j(\varepsilon)$ are unitarily equivalent to $T^{s_1}(\varepsilon)$ and the $T_k^*(\varepsilon)$ are unitarily equivalent to $T^{s_2}(\varepsilon)$. Moreover, by corollary 1 there exist constants a and r' depending only on s_1 and s_2 (since all pairs $(|f_j|, |g_k|)$ are unitarily equivalent to $(|z_1 - (1 - s_1)|, |z_2 - (1 - s_2)|)$) such that

$$\{|f_j g_k| \leq a\varepsilon\} \cap r\mathbb{B}^2 \subset \left((\{|f_j| \leq \varepsilon\} \cup \widehat{\{|g_k| \leq \varepsilon\}}) \cap r\partial\mathbb{B}^2 \right)$$

for all $r \in [r', 1]$ and all j and k. The left hand side contains the bidisc $\{|f_j| \leq \sqrt{a\varepsilon}\} \cap \{|g_k| \leq \sqrt{a\varepsilon}\} = (\zeta_1^j, \zeta_2^k) + \sqrt{a\varepsilon}\,(\mathbb{D} \times \mathbb{D})$ of radius $\sqrt{a}\sqrt{\varepsilon}$ around the intersection point $\ell_{f_j} \cap \ell_{g_k} = (\zeta_1^j, \zeta_2^k)$ (with r' close to 1 and ε small enough). If $\sqrt{a}\sqrt{\varepsilon} > (B+1)\varepsilon$ we obtain (running over all pairs (j, k)) that the right hand side of (4_r) contains the product of the circles $qR_1\mathbb{T}$ and $qR_2\mathbb{T}$, hence (4_r) holds.

Finally, taking from the beginning q close to 1 (and $\varepsilon > 0$ small), we ensure that all $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are contained in a small neighbourhood of $\partial\mathcal{D}_1 \times \partial\mathcal{D}_2$. $\qquad\square$

The following proposition allows to cover small enough neighbourhoods of given complex affine discs (intersections of complex lines with suitable balls) by polynomial hulls of unions of disjoint tori with small diameter contained in the sphere. Propositions 2, 3 and 4 below will be stated for tori related to the function $z_1 - (1 - s)$ for some $s \in (0, 1)$. They hold for $z_1 - (1 - s)$ replaced by any complex affine function f such that $|f|$ is unitarily equivalent to $|z_1 - (1 - s)|$.

Proposition 2. *Let $s \in (0, 1)$. For any small $\delta > 0$, any $q \in (0, 1)$ close to 1 and any small $s' > 0$ one can find two positive numbers $s_1 < s'$, $s_2 < s'$ and a positive number $\varepsilon' = \varepsilon'(s, s_1, s_2, \delta)$ such that for any positive $\varepsilon < \varepsilon'$ there exist two finite families of complex affine functions f_j and g_k with the following properties:*
The related solid tori $T_j(2\varepsilon) \stackrel{\text{def}}{=} T_{f_j}(2\varepsilon)$ and $T_k^*(2\varepsilon) \stackrel{\text{def}}{=} T_{g_k}(2\varepsilon)$ are all pairwise disjoint and contained in $\{|z_1 - (1 - s)| \leq \delta\}$. The $T_j(\varepsilon)$ are unitarily equivalent to $T^{s_1}(\varepsilon)$ and the $T_k^*(\varepsilon)$ are unitarily equivalent to $T^{s_2}(\varepsilon)$. Moreover

$$(5) \qquad \overline{q\mathbb{B}^2} \cap \{|z_1 - (1 - s)| \leq \varepsilon\} \subset \left(\widehat{\bigcup_j T_j(\varepsilon) \cup \bigcup_k T_k^*(\varepsilon)} \right).$$

The following stronger assertion holds. There exists $r' < 1$ such that for any $r \in [r', 1]$ the inclusion

$$(5_r) \qquad \overline{q\mathbb{B}^2} \cap \{|z_1 - (1 - s)| \leq \varepsilon\} \subset \left(\widehat{\bigcup_j {}^r T_j(\varepsilon) \cup \bigcup_k {}^r T_k^*(\varepsilon)} \right).$$

holds.

Note that the number of tori and their symmetry axes will be chosen together with ε.

Remark 3. *Proposition 2 describes a subset of the sphere whose connected components have a priori given small diameter such that its polynomial hull contains e.g. the disc $\{z_1 = 0, \ |z_2| \leq q\}$ $(q < 1)$. This statement has to be opposed to the following still open conjecture of Vitushkin which appeared in connection with a problem of E. Kallin on polynomial convexity of finite unions of disjoint balls.*

Conjecture (Vitushkin). *Let X be a relatively closed complex manifold of dimension one in $\mathbb{B}^2$ smooth up to the boundary and transversal to $\partial\mathbb{B}^2$. Suppose X contains the origin. Then X has a boundary component of diameter bounded from below by an absolute constant.*

The link between Vitushkin's conjecture and the above statement (or the theorem, respectively) is the following open problem.

Problem. *Let $K \subset \mathbb{C}^2$ be a compact, $K \neq \hat{K}$. Suppose $z \in \hat{K} \setminus K$. Under which conditions for every $\varepsilon > 0$ the ε-neighbourhood K_ε of K contains the boundary of a Riemann surface passing through z?*

Note that proposition 2 does not control the ratio of the numbers ε (the width of the tubular neighbourhood of ℓ^s which is covered by the polynomial hull of the union of small tori) and δ (the width of the tubular neighbourhood which contains the small tori). Propositions 3 and 4 below will allow such a control, after which the inductive construction of the E_N can be easily provided.

Plan of proof of proposition 2. The functions f_j in proposition 2 will be chosen so that ℓ_{f_j} will pass through a point p_j where the p_j are equidistributed on the circle $\{z_1 = 1 - s\} \cap R\partial\mathbb{B}^2$ for a suitable $R < 1$ and close to 1, and the ℓ_{f_j} are complex tangent to the sphere $R\partial\mathbb{B}^2$. The ℓ_{g_j} pass through points p_j^* which are equidistributed on the same circle. Moreover, the p_j^* are obtained from p_j by turning by a fixed angle ψ in z_2-direction and the ℓ_{g_j} are obtained by turning the complex tangents to the sphere $R\partial\mathbb{B}^2$ at p_j^* by a fixed angle ν. Lemma 2 below states that for suitable p_j, R and ε the tubes $\{|f_j| \leq 2\varepsilon\} \cap \overline{\mathbb{B}^2}$ are pairwise disjoint. (Hence the tori $T_j(2\varepsilon) = \{|f_j| \leq 2\varepsilon\} \cap \partial\mathbb{B}^2$ are pairwise disjoint and not linked with each other in $\partial\mathbb{B}^2$). Moreover, with a suitable choice of the angle ν the same is true for the tubes $\{|g_k| \leq 2\varepsilon\} \cap \overline{\mathbb{B}^2}$ and the tori $T_k^*(2\varepsilon)$.

On the other hand each complex line ℓ_{f_j} intersects several lines ℓ_{g_k} inside $\mathbb{B}^2$ (equivalently, the corresponding circles $\ell_{f_j} \cap \partial\mathbb{B}^2$ and $\ell_{g_k} \cap \partial\mathbb{B}^2$ are linked in $\partial\mathbb{B}^2$). Lemma 3 below allows to choose the angle ψ in such a way that the tubes $\{|f_j| \leq 2\varepsilon\}$ and $\{|g_k| \leq 2\varepsilon\}$ intersect at points which are not contained in the sphere $\partial\mathbb{B}^2$, i.e. the tori $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are disjoint. An application of the main lemma will give the proposition.

The tori $T_j(\varepsilon)$ and $T_j^*(\varepsilon)$. The $T_j(\varepsilon)$ will be determined by the following parameters: $s \in (0,1)$, a small number $t > 0$, a (large) natural number $\mathcal{N}$ and a small number $\varepsilon > 0$. The $T_j^*(\varepsilon)$ will be determined by s, t, $\mathcal{N}$, ε and a small number ν.

Consider the intersection

$$\ell^s \cap \partial \mathbb{B}^2 = \{(1 - s, R_2 e^{i\phi}); \phi \in [0, 2\pi)\} \quad (R_2^2 = 2s - s^2).$$

Denote by C_t the slightly smaller concentric circle $C_t = \{(1 - s, R_2(1 - t)e^{i\phi}) : \phi \in [0, 2\pi)\} = \ell^s \cap R\partial\mathbb{B}^2$, where R and t are related by the equality $(2t - t^2)(2s - s^2) = 1 - R^2$.

Let p_j be equidistributed points on C_t,

$$p_j = (1 - s, R_2(1 - t)e^{i\phi_j}), \quad \text{where } \phi_j = \frac{2\pi}{\mathcal{N}}j, \quad j = 0, \ldots, \mathcal{N} - 1.$$

Define the constant B by the following relation: the distance between nearest of the equidistributed points, $|p_j - p_{j-1}|$, equals $B \cdot \varepsilon$. Denote by ℓ_j the complex lines through p_j which are tangent to $R\partial\mathbb{B}^2$,

(6) $$\ell_j = \{p_j + v_j \cdot \zeta : \zeta \in \mathbb{C}\}, \quad v_j = ((1 - t)R_2, -(1 - s)e^{i\phi_j}),$$

and let f_j be annihilating functions of the ℓ_j with gradient of norm one.

For a (small) number ψ denote by p_j^* the points

$$p_j^* = \left(1 - s, R_2(1 - t)e^{i(\phi_j + \psi)}\right).$$

Finally, for a small constant $\nu > 0$ we denote by ℓ_j^*, $j = 0, \ldots, \mathcal{N} - 1$, the complex line obtained from the complex tangent to $R\partial\mathbb{B}^2$ at p_j^* by turning by a fixed angle,

(7) $$\ell_j^* = \{p_j^* + w_j \cdot \zeta : \zeta \in \mathbb{C}\}, \quad w_j = \left((1 - t)R_2, -(1 - s + \nu)e^{i(\phi_j + \psi)}\right).$$

Let g_j be annihilating functions of ℓ_j^* with gradient of norm one.

Lemma 2. *a) Let $s \in (0, 1)$ and $t > 0$ be sufficiently small. There exist positive constants $\varepsilon' = \varepsilon'(s, t)$ and $B' = B'(s, t)$ such that if $\varepsilon \in (0, \varepsilon')$ and $B > B'$ then for the f_j defined above for chosen parameters s, t, $\mathcal{N}$ and ε (with B related to $\mathcal{N}$ and ε as above) the sets $\{|f_j| \leq 2\varepsilon\} \cap \overline{\mathbb{B}^2}$ are pairwise disjoint.*
b) If in addition ν is small enough (depending on s and t) there exist constants $\varepsilon' = \varepsilon'(s, t, \nu)$ and $B' = B'(s, t, \nu)$ such that if $\varepsilon \in (0, \varepsilon')$ and $B > B'$ then for the g_j defined for the parameters s, t, $\mathcal{N}$, ν and ε and an arbitrary parameter ψ the sets $\{|g_j| \leq 2\varepsilon\} \cap \overline{\mathbb{B}^2}$ are pairwise disjoint.

Lemma 3. *With suitable constants ε' and B' which are greater than the constants of lemma 2a and 2b and parameters s, t, $\mathcal{N}$, ν and ε as in lemma 2 with $B > B'$ one can choose the constant ψ in such a way that the tori $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ of lemma 2 are disjoint for all j and k.*

Proof of lemma 2. The argument is roughly, that for small $t > 0$ the intersection $\ell_j \cap \overline{\mathbb{B}^2}$ is a disc of small diameter. If two such discs intersect their centers p_j must be close. But then the corresponding complex tangencies to $R\partial\mathbb{B}^2$ are "almost parallel" so they cannot intersect at points close to the p_j.

More precisely, the set $\{|f_j| \leq 2\varepsilon\}$ is the union of complex lines $\mathcal{L}_j = \{\tilde{p}_j + v_j \cdot \zeta : \zeta \in \mathbb{C}\}$ with the same direction v_j as ℓ_j through points $\tilde{p}_j \in \ell^s$ with distance from p_j not exceeding $A\varepsilon$ for a constant A depending on the angle between ℓ^s and ℓ_j, hence by unitary equivalence, on s and t only. Hence,

$$(8) \qquad \tilde{p}_j = (1 - s, R_2(1 - t)e^{i\phi_j} + \alpha_j) \text{ with } |\alpha_j| \leq A\varepsilon.$$

Let $j \neq k$. For the intersection point $\mathcal{L}_j \cap \mathcal{L}_k \neq \emptyset$ we have

$$\tilde{p}_j + v_j\zeta = \tilde{p}_k + v_k\zeta' \text{ for some } \zeta, \zeta' \in \mathbb{C}.$$

From (6) and (8) we obtain that $\zeta = \zeta'$ and

$$(9) \qquad \zeta = \frac{R_2(1 - t)}{(1 - s)} + \frac{\alpha_j - \alpha_k}{(1 - s)(e^{i\phi_j} - e^{i\phi_k})} \text{ with } |\alpha_j| \leq A\varepsilon, \ |\alpha_k| \leq A\varepsilon.$$

For small $t > 0$ and $\varepsilon < \varepsilon'(t, s)$ the absolute value $|\zeta|$ of this number can be estimated from below by a positive constant depending only on s, provided $B \geq B'(s, t)$ and $B'(s, t)$ is chosen so that

$$|e^{i\phi_j} - e^{i\phi_k}|^{-1} \cdot 2A\varepsilon \leq \frac{2A\varepsilon}{B'(s, t) \cdot \varepsilon} < \frac{1}{2}R_2(1 - t).$$

On the other hand $\ell_j \cap \overline{\mathbb{B}^2}$ is a closed disc of radius $\sqrt{1 - R^2}$ which, for fixed s, is small if $t > 0$ is small. Hence for $\varepsilon < \varepsilon'(t, s)$ the diameter of the intersection $\mathcal{L}_j \cap \overline{\mathbb{B}^2}$ is small ($\mathcal{L}_j$ as above contained in $\{|f_j| \leq 2\varepsilon\}$). For those t and ε the intersection $\mathcal{L}_j \cap \overline{\mathbb{B}^2}$ cannot contain the two points $\tilde{p}_j$ and $\tilde{p}_j + v_j\zeta = \mathcal{L}_j \cap \mathcal{L}_k$ since their distance is bounded from below by a constant depending only on s. Since $\tilde{p}_j \in \mathcal{L}_j \cap \overline{\mathbb{B}^2}$ the point $\tilde{p}_j + v_j\zeta$ is not in $\overline{\mathbb{B}^2}$. Part a) is proved.

To prove assertion b) increase A if necessary and replace the number $(1 - s)$ in (9) by $(1 - s + \nu)$. Use that for ν small, $\nu < \nu(s, t)$, and $\varepsilon < \varepsilon'(s, t, \nu)$, the complex lines $\mathcal{L}_j^*$ parallel to ℓ_j^* and of distance not exceeding 2ε from ℓ_j^* still intersect $\overline{\mathbb{B}^2}$ along a disc of small diameter. The remaining arguments are the same as for part a). $\quad\square$

Remark 4. *Note that the unitary transformation $(z_1, z_2) \to (z_1, z_2 e^{-i\phi_j})$ maps the torus $T_j(\varepsilon) = T_{f_j}(\varepsilon)$ to the torus $T_0(\varepsilon) = T_{f_0}(\varepsilon)$ and $T_j^*(\varepsilon) = T_{g_j}(\varepsilon)$ to $T_0^*(\varepsilon) = T_{g_0}(\varepsilon)$. Moreover, the tori $T_j(\varepsilon)$ are unitarily equivalent to $T^{s_1}(\varepsilon)$ with*

$$(10) \qquad s_1 = 1 - R = (1 + R)^{-1}(2t - t^2)(2s - s^2)$$

and $T_j^*(\varepsilon)$ are unitarily equivalent to $T^{s_2}(\varepsilon)$ for some number $s_2 > s_1$ which is close to s_1 if ν is small.

Proof of lemma 3. We want to choose ψ so that $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are disjoint for all j and k. Note that the norm $|P_{j,k}|$ of the intersection point $P_{jk} = \ell_j \cap \ell_k^*$ depends only on $m = k - j$. The idea is the following. When $|P_{0,m}|$ is close to 1, the points $P_{0,m}$ form approximately an arithmetic progression with step $const \cdot B\varepsilon$ on a real line in the complex line ℓ_0. Changing the parameter ψ leads approximately to translating the

approximate arithmetic progression by the parameter ψ inside the real line. If B is large enough, this allows to choose ψ in such a way that the intersection points $P_{0,m}$ are not in $\partial \mathbb{B}^2$, moreover, a neighbourhood of them of size comparable with ε (containing the intersection $\{|f_0| \le 2\varepsilon\} \cap \{|g_m| \le 2\varepsilon\}$) does not meet $\partial \mathbb{B}^2$.

Here is the precise argument. Let s, t, $\mathcal{N}$, ν and ε be chosen according to lemma 2 with constants ε' and B' specified below and greater than the constants in part a and b of the lemma. We will change the parameter ψ. The points p_k^*, the complex lines ℓ_k^*, the intersection points $P_{jk} = \ell_j \cap \ell_k^*$ and the tori $T_k^*(\varepsilon)$ will depend on ψ. We will indicate the dependence on ψ only sometimes when we want to draw special attention to this fact.

From (6) and (7) it follows that the intersection point $P_{j,k} = P_{j,k}(\psi)$ of the complex lines ℓ_j and ℓ_k^* is determined by

$$P_{j,k}(\psi) = p_j + v_j \cdot \zeta(\psi) = p_k^*(\psi) + w_k(\psi) \cdot \zeta'(\psi)$$

for some $\zeta(\psi), \zeta'(\psi) \in \mathbb{C}$. From the same formulas we obtain that $\zeta(\psi) = \zeta'(\psi) \overset{\text{def}}{=} \zeta_{j,k}(\psi)$ and

$$(11) \qquad \zeta_{j,k}(\psi) = \frac{R_2 \cdot (1-t)}{1-s} \frac{1 - e^{i(\phi_k - \phi_j + \psi)}}{1 - Q e^{i(\phi_k - \phi_j + \psi)}},$$

where

$$Q = (1 - s + \nu)(1 - s)^{-1} > 1.$$

Here as above we put $\phi_j = \frac{2\pi j}{\mathcal{N}}$ and assume that the natural number $\mathcal{N}$ is big. Put

$$(12) \qquad F(\phi) = \left| \frac{1 - e^{i\phi}}{1 - Q e^{i\phi}} \right| = \left| \frac{2 \sin \frac{\phi}{2}}{1 - Q e^{i\phi}} \right|,$$

so that

$$(13) \qquad |\zeta_{j,k}(\psi)| = \frac{R_2(1-t)}{1-s} F(\phi_k - \phi_j + \psi).$$

Note that the function $(|1 - Q e^{i\phi}|)^{-1}$ is of class $C^\infty(\mathbb{R})$, hence for small $\phi \ne 0$ we have

$$F'(\phi) = \frac{\cos \frac{\phi}{2} \cdot \operatorname{sgn}\phi}{|1 - Q e^{i\phi}|} + |2 \sin \frac{\phi}{2}| \cdot \left(\frac{1}{|1 - Q e^{i\phi}|} \right)'.$$

Hence

$$(14) \qquad 0 < C_Q \le |F'(\phi)| \le 2C_Q \text{ for } \phi \ne 0 \text{ and } |\phi| \le \Phi_Q$$

for positive constants Φ_Q and C_Q which may be chosen depending only on Q. Since $p_j \in R\partial \mathbb{B}^2$ and v_j is the direction of the complex tangent line to $R\partial \mathbb{B}^2$ we have

$$|P_{j,k}|^2 = |p_j + v_j \cdot \zeta_{j,k}|^2 = |p_j|^2 + |v_j|^2 |\zeta_{j,k}|^2.$$

By (6) $|v_j|^2 = |p_j|^2 = R^2$. Hence

$$(15) \qquad |P_{j,k}(\psi)|^2 = R^2 \left(1 + \frac{R_2^2 (1-t)^2}{(1-s)^2} F^2(\phi_k - \phi_j + \psi) \right),$$

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with

(16) $$R^2 = 1 - (2t - t^2)(2s - s^2).$$

The set $\{|f_j| \leq 2\varepsilon\} \cap \{|g_k| \leq 2\varepsilon\}$ is contained in the (closed) $A'\varepsilon$-neighbourhood of $P_{j,k} = P_{j,k}(\psi)$. The constant A' depends only on ν, s and t. For the square of the norm of points in $\{|f_j| \leq 2\varepsilon\} \cap \{|g_k| \leq 2\varepsilon\}$ this implies that these numbers are contained in the (open) $4A'\varepsilon$-neighbourhood of $|P_{j,k}(\psi)|^2$ provided $|P_{j,k}(\psi)| \leq \frac{3}{2}$ and $A'\varepsilon < 1$.

We want to choose ψ so that the $4A'\varepsilon$-neighbourhoods of the $|P_{j,k}(\psi)|^2$ do not contain 1. Put $m = k - j$. Then $\phi_k - \phi_j + \psi = \phi_m + \psi = -\phi_{-m} + \psi$ and $|P_{j,k}(\psi)|^2 = |P_{0,m}(\psi)|^2 = |P_{0,-m}(-\psi)|^2$. Hence all possible values of (15) are obtained when $j = 0$ and $k = m$ runs over integers.

If for some ψ and m_0

(17) $$\left| |P_{0,m_0}(\psi)|^2 - 1 \right| < 4A'\varepsilon,$$

then by (15) and (16) $F^2(\phi_{m_0} + \psi) \, t^{-1}$ can be estimated from above and from below by positive constants depending only on s provided ε is small compared with t. Hence in this case $\phi_{m_0} + \psi$ is comparable to either $\sqrt{t}$ or $-\sqrt{t}$.

Suppose now that for for some m_0 the $4A'\varepsilon$-neighbourhood of $|P_{0,m_0}(0)|^2$ contains 1. (Otherwise we are done.) Note that by symmetry this is true also for $-m_0$. If t is small then by the preceding arguments ϕ_{m_0} and ϕ_{-m_0} are much smaller than Φ_Q. By (14) and (15) for fixed ψ close to 0 and for integer numbers m close to m_0 (close to $-m_0$, respectively) the possible values of the right hand side of (15) have distance from each other at least equal to $c(s,t) \cdot R_2(1-t)|\phi_{m+1} - \phi_m|$ for a positive constant $c(s,t)$ depending on s and t. Since $|\phi_{m+1} - \phi_m| = \frac{2\pi}{\mathcal{N}} > 2\sin\frac{\pi}{\mathcal{N}}$ and $R_2(1-t)2\sin\frac{\pi}{\mathcal{N}} = B\varepsilon$, the difference of the values of (15) for fixed ψ close to 0 and for m close to m_0 (close to $-m_0$, respectively) is at least $c(s,t) \cdot B \cdot \varepsilon$. Take the constant B' large enough so that the latter constant is at least $40A'\varepsilon$.

Using again (14) and (15) take ψ so that

$$10A'\varepsilon < ||P_{0,m_0}(\psi)|^2 - |P_{0,m_0}(0)|^2| < 30A'\varepsilon$$

and the same estimate holds for m_0 replaced by $-m_0$. Note that ψ is comparable to ε with multiplicative constants depending on s and t and, hence, ψ is small if ε is small. The $4A'\varepsilon$-neighbourhood of $|P_{0,m_0}(\psi)|^2$ does not contains 1 and the same is true for m_0 replaced by $-m_0$. The above arguments give that for all m close to m_0

$$||P_{0,m}(\psi)|^2 - 1| > 4A'\varepsilon,$$

hence, since the function F is strictly monotonic on the positive half-axis (respectively, on the negative half-axis) this holds for all m.

With this choice of ψ we obtained that for all j and k the set $\{|f_j| \leq 2\varepsilon\} \cap \{|g_k| \leq 2\varepsilon\}$ does not meet $\partial\mathbb{B}^2$ if ε is small enough, hence the tori $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are disjoint. $\square$

Proof of proposition 2. For $s \in (0,1)$ as in the statement we first choose t and ν small enough, so that the numbers s_1 and s_2 are less than s' (see (10) and remark 4) and

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$R(t) > q$ (see (16)). We may also achieve by choosing t and ν small that with any choice of the natural number $\mathcal{N}$, of the parameter ψ and of the small enough positive number ε the sets $\{|f_j| \leq 2\varepsilon\}$ and $\{|g_k| \leq 2\varepsilon\}$ are contained in $\{|z_1 - (1-s)| \leq \delta\}$. Choose by lemmas 2 and 3 the relation of the numbers $\mathcal{N}$, ε and ψ so that the tori $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are pairwise disjoint. They are unitarily equivalent to $T^{s_1}(\varepsilon)$ and $T^{s_2}(\varepsilon)$, respectively.

It remains to prove (5). Apply the main lemma to the pair ℓ_j and $\ell_j^* = \ell_j^*(\psi)$ for any j. Since the pairs obtained for different j are unitarily equivalent to ℓ_0 and $\ell_0^*(\psi)$ (see remark 4) and $\ell_0^*(\psi)$ is close to $\ell_0^*(0)$ if ψ is small, there exist numbers $a > 0$ and $r' \in (0,1)$ depending only on s, t and ν such that (1) holds with f replaced by f_j and g replaced by g_j. By corollary 1 for $r \in [r', 1]$ the polynomial hull of ${}^r T_j(\varepsilon) \cup {}^r T_j^*(\varepsilon)$ contains

$$\{|f_j g_j| \leq a\varepsilon\} \cap \overline{r\mathbb{B}^2} \supset \{|f_j| \leq \sqrt{a\varepsilon}\} \cap \{|g_j| \leq \sqrt{a\varepsilon}\} \cap \overline{r\mathbb{B}^2}.$$

Since $f_j(p_j) = 0$ and $g_j(p_j^*) = 0$, we obtain $|g_j(p_j)| = |g_j(p_j) - g_j(p_j^*)| \leq |p_j - p_j^*| = O(\varepsilon)$. This implies that for small ε the latter set contains the bidisc $p_j + b\sqrt{a\varepsilon}\,(\overline{\mathbb{D}} \times \overline{\mathbb{D}})$ of radius $b \cdot \sqrt{a\varepsilon}$ around the point p_j for some constant b depending only on s, t and ν. Recall that the p_j are equidistributed on the circle $C_t = \{1-s\} \times \{|z| = R_2(1-t)\}$ with distance between nearest points equal to $B\varepsilon$. If $b \cdot \sqrt{a\varepsilon} > B\varepsilon$, i.e. $\varepsilon < \frac{b^2}{B^2}a$, then the polynomial hull of $\bigcup_j (T_j(\varepsilon) \cup T_j^*(\varepsilon))$ contains $\{z_1\} \times \{|z| = R_2(1-t)\}$ for all z_1 with $|z_1 - (1-s)| \leq B\varepsilon$. Since $(1-s)^2 + (R_2(1-t))^2 = R(t)^2 > q^2$ we obtain that for $\varepsilon < \varepsilon'$ with a suitable choice of the constants r' and $\varepsilon'(s, s_1, s_2, \delta)$ the inclusion (5_r) holds for $r \in [r', 1]$. The weaker inclusion (5) follows. $\qquad\square$

The proof of proposition 2 does not give good estimates for the ratio of the constants ε and δ. The point is that the main lemma gives useful effects essentially only for the pair ℓ_j and ℓ_j^* of complex lines but not for arbitrary pairs ℓ_j and ℓ_k^* when the intersection point may be close to $\partial\mathbb{B}^2$ and the constant a of the main lemma is not bounded away from zero. Below we will state and prove the stronger proposition 4 which can be directly used for inductive construction of the E_N.

The first step towards this goal is proposition 3 below which is in the spirit of proposition 1. For its proof we use again two families of complex lines. The first family, ℓ_k^*, consists of certain complex lines parallel to the z_2-axis. For some small $\sigma > 0$ and a suitable constant α their z_1-coordinates form an α-net of the $\frac{4}{3}\sigma$-neighbourhood of $1 - s$ ($s \in (0,1)$). Note that the intersection of these lines with $\partial\mathbb{B}^2$ are circles of diameters comparable with $\sqrt{s}$. The second family, ℓ_j, consists of complex tangent lines to a smaller sphere through equidistributed points on a circle C_t contained in $\ell^s = \{z \in \mathbb{C}^2 : z_1 = 1 - s\}$. This time t cannot be chosen arbitrarily small. It will take a value which is determined by s and σ. As in proposition 1 the main lemma will be applied to all pairs (ℓ_j, ℓ_k^*).

Proposition 3. *Let $s \in (0,1)$. There exist positive constants $\sigma' = \sigma'(s)$ and $\alpha = \alpha(s, \sigma')$ such that the following holds.*
Fix $\sigma \in (0, \sigma')$. Let ζ_k be all points of the disc $\{\zeta \in \mathbb{C} : |\zeta - s| \leq \frac{4}{3}\sigma\}$ which are contained in the lattice $s + \alpha\mathbb{Z} + i\alpha\mathbb{Z}$. Denote $g_k(z) = z_1 - (1 - \zeta_k)$.
There exist positive constants $\varepsilon' = \varepsilon'(s, \sigma)$ and $s^* = s^*(s, \sigma) < s$, $s^*(s, \sigma) \to 0$ for $\sigma \to 0$, such that for any $\varepsilon \in (0, \varepsilon')$ the following is true.

One can find finitely many complex affine functions f_j such that the tori $T_j(\varepsilon) = \{z \in \partial\mathbb{B}^2 : |f_j(z)| \leq \varepsilon\}$ are unitarily equivalent to $T^{s^*}(\varepsilon)$. Moreover, the tori $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon) = \{z \in \partial\mathbb{B}^2; |g_k(z)| \leq 2\varepsilon\}$ are all pairwise disjoint and contained in $\{|z_1 - (1-s)| \leq 2\sigma\}$. Finally, for the polynomial hull of the union of the tori we have

$$(18) \qquad \{|z_1 - (1-s)| \leq \sigma\} \cap (1-\rho)\overline{\mathbb{B}^2} \subset \left(\widehat{\bigcup_j T_j(\varepsilon) \cup \bigcup_k T_k^*(\varepsilon)} \right)$$

for a constant $\rho = \rho(s,\sigma)$ such that $\rho(s,\sigma) \to 0$ for s fixed and $\sigma \to 0$.
The following sharper statement holds.
There exists a constant $r' = r'(s,\sigma) > 0$ such that for $r \in [r', 1]$

$$(18_r) \qquad \{|z_1 - (1-s)| \leq \sigma\} \cap (1-\rho)\overline{\mathbb{B}^2} \subset \left(\widehat{\bigcup_j {}^r T_j(\varepsilon) \cup \bigcup_k {}^r T_k^*(\varepsilon)} \right).$$

Proof. Let s and σ be given as in the statement of the proposition. The complex lines ℓ_j will be obtained as in the plan of proof of proposition 2 for some parameters t and $\mathcal{N}$: they will be complex tangent to the sphere $R(t)\partial\mathbb{B}^2$ (see (16)) through equidistributed points $p_j \in C_t$ with distance $|p_{j+1} - p_j|$ depending on $\mathcal{N}$.

Determine now the parameter t (so far $\mathcal{N}$ is arbitrary). Let t be maximal so that $\ell_j \cap \partial\mathbb{B}^2 \subset \{|z_1 - (1-s)| \leq \frac{5}{3}\sigma\}$ and, hence, for this parameter t $\ell_j \cap \partial\mathbb{B}^2 \subset \{|z_1 - (1-s)| = \frac{5}{3}\sigma\}$ (see (6)). This number t does not depend on $\mathcal{N}$ and j and tends to zero for $\sigma \to 0$. Hence, also $s^*(s,\sigma) \to 0$ for $\sigma \to 0$. Moreover, if σ' is small, then there is a uniform estimate from below of the angle of intersection of $\ell^{s'}$ and ℓ_j for $s' \in (s - 2\sigma, s + 2\sigma)$, $\sigma \in (0, \sigma')$, t related to σ as just described. Indeed, $\ell^{s'}$ is transversal to the complex tangent space of $\partial\mathbb{B}^2$ with a uniform estimate of the angle for the mentioned s'. Hence, if σ' is small and $\sigma < \sigma'$ then t is small, hence $R(t)$ is close to one, and, hence, the ℓ_j (being complex tangents to $R(t)\partial\mathbb{B}^2$) are transversal to the $\ell^{s'}$ with uniform estimate of the angle.

The positive number α will be specified later. Let ζ_k, g_k and $\ell_k^* = \{g_k = 0\}$ be as in the statement of the proposition. The ℓ_k^* are parallel to ℓ^s and $\ell_k^* \cap \partial\mathbb{B}^2$ is contained in $\{|z_1 - (1-s)| \leq \frac{4}{3}\sigma\}$ while $\ell_j \cap \partial\mathbb{B}^2$ is contained in $\{|z_1 - (1-s)| = \frac{5}{3}\sigma\}$. The above observations imply two facts.

First, if σ' and hence t are small and ε is small depending on s' and σ and, moreover, the parameters $\mathcal{N}$ and ε satisfy the conditions of lemma 2a, then all tori $T_j(2\varepsilon)$ and $T_k^*(2\varepsilon)$ are disjoint and contained in $\{|z_1 - (1-s)| \leq 2\sigma\}$.

Secondly, the main lemma can be applied to all pairs (ℓ_j, ℓ_k^*) with uniform constants a and r'.

Moreover, using the notation $\ell^\xi = \{z \in \mathbb{C}^2 : z_1 = 1 - \xi\}$ with $\xi \in \mathbb{C}$, $|\xi - s| \leq \frac{4}{3}\sigma$, we obtain that for each positive number η the set $\ell^\xi \cap \{|f_j| \leq \eta\}$ is a closed disc in ℓ^ξ of radius at least $D \cdot \eta$ around the intersection point $\ell^\xi \cap \ell_j$ with constant D depending only on s and σ'.

Prove now the assertion on the polynomial hull of the union of the tori for suitably chosen α. Note first that for $|\xi - s| \leq \frac{4}{3}\sigma$ the intersection points $\ell^\xi \cap \ell_j$, $j = 0, \ldots, \mathcal{N} - 1$, are equidistributed on the circle $\{1 - \xi\} \times \{|z| = R_\xi'\} \subset \mathbb{B}^2$ for a number R_ξ' which is close to $\sqrt{1 - |1 - \xi|^2}$ if $\sigma < \sigma'$ is small. This follows from the fact that the unitary transformation $(z_1, z_2) \to (z_1, e^{i\phi_j} z_2)$ maps the pair (ℓ^ξ, ℓ_0) to the pair (ℓ^ξ, ℓ_j).

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Corollary 1 implies that for $r \in [r', 1]$ and all k and j the polynomial hull of ${}^r T_k^*(\varepsilon) \cup {}^r T_j(\varepsilon)$ contains the set

(19) $$\{|z_1 - (1 - \zeta_k)| \cdot |f_j(z)| \le a\varepsilon\} \cap r\overline{\mathbb{B}^2}.$$

Let ε be as small as required above, i.e. $\varepsilon < \varepsilon'(s, \sigma)$, and let $B > B'$ with B' the constant of lemma 2a. Choose $\mathcal{N}$ so that the distance between nearest points $|p_{j+1} - p_j|$ is between $B\varepsilon$ and $(B + 1)\varepsilon$. If σ is small then for $|\xi - s| \le \frac{4}{3}\sigma$ the distance between nearest of intersection points $\ell^\xi \cap \ell_j$, $j = 0, \ldots, \mathcal{N} - 1$, does not exceed $2(B + 1)\varepsilon$. Take a constant C so that $C \cdot D > 2(B + 1)$. The set in (19) contains the set

$$\{z \in \overline{r\mathbb{B}^2} : |z_1 - (1 - \zeta_k)| \le \frac{a}{C}, \ |f_j(z)| \le C\varepsilon\}.$$

Hence, for any fixed ξ with $|\xi - \zeta_k| \le \frac{a}{C}$ and for r as above the polynomial hull of ${}^r T_k^*(\varepsilon) \cup {}^r \bigcup_j T_j(\varepsilon)$ contains the circle $\{1 - \xi\} \times \{|z| = R'_\xi\}$. Take any $\alpha < \frac{a}{C}$. Then for $r \in [r', 1]$

$$\left(\bigcup_k \widehat{{}^r T_k^*(\varepsilon) \cup \bigcup_j {}^r T_j(\varepsilon)} \right) \supset \bigcup_{|\xi - s| \le \sigma} \{1 - \xi\} \times \{|z| \le R'_\xi\}.$$

The right hand side contains $\{|z_1 - (1 - s)| \le \sigma\} \cap (1 - \rho)\overline{\mathbb{B}^2}$ for suitable $\rho = \rho(s, \sigma)$ with $\rho(s, \sigma) \to 0$ for $\sigma \to 0$. $\square$

Now we are ready to state and prove the main proposition.

Proposition 4. *Let $s \in (0, 1)$. There exists a positive constant $\sigma' = \sigma'(s)$ such that for any $\sigma \in (0, \sigma')$ there exist finitely many numbers $s_m \in (0, 1)$ (their number depends on s and σ and each of them tends to zero for $\sigma \to 0$) and a positive number $\varepsilon' = \varepsilon'(s, \sigma, s_m)$ such that the following holds.*
For any $\varepsilon \in (0, \varepsilon')$ there exist finitely many complex affine functions f_n and related tori $T_n(2\varepsilon)$, each of them unitarily equivalent to $T^{s_m}(2\varepsilon)$ for some m with $T_n(2\varepsilon)$ pairwise disjoint and contained in $\{|z_1 - (1 - s)| \le 2\sigma\}$ and such that

(20) $$(1 - \rho)\overline{\mathbb{B}^2} \cap \{|z_1 - (1 - s)| \le \sigma\} \subset \left(\widehat{\bigcup_n T_n(\varepsilon)} \right)$$

for a constant $\rho = \rho(s, \sigma)$ such that for fixed s $\rho(s, \sigma) \to 0$ if $\sigma \to 0$. Moreover, there exists $r' = r'(s, \sigma, s_m)$ such that for $r \in [r', 1]$

(20$_r$) $$(1 - \rho)\overline{\mathbb{B}^2} \cap \{|z_1 - (1 - s)| \le \sigma\} \subset (\widehat{\bigcup_n {}^r T_n(\varepsilon)}).$$

Proof. Let σ', r' and α be as in proposition 3, let $\sigma < \sigma'$ and let $\varepsilon'(s, \sigma)$ be the constant from the statement of proposition 3. Proposition 3 gives two families of functions f_j and g_k. If σ is small the tori related to the f_j have small diameter, but the tori related to the g_k have large diameter. Our aim is to apply proposition 2 to each g_k.

Let $s(g_k)$ be the number for which $|g_k|$ is unitarily equivalent to $|z_1 - (1 - s(g_k))|$. Apply for each k proposition 2 with $s = s(g_k)$, $\delta = \frac{\alpha}{3}$ and $q = r'$. For each k we obtain

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two numbers $s_{k,1}, s_{k,2} \in (0,1)$ (it can be achieved that they are as small as we wish) and a bound for ε below of which we can find finitely many non-intersecting tori $T(2\varepsilon)$, unitarily equivalent to $T^{s_{k,1}}(2\varepsilon)$ or $T^{s_{k,2}}(2\varepsilon)$, contained in $\{|z_1 - (1 - s(g_k))| \leq \frac{\alpha}{3}\}$ with the following property

$$(21_r) \qquad \qquad \overline{r'\mathbb{B}^2} \cap \{|z_1 - (1 - s(g_k))| \leq \varepsilon\} \subset (\widehat{\bigcup {}^r T(\varepsilon)})$$

for all $r \in [\tilde{r}_k, 1]$ with a suitable $\tilde{r}_k$.

Assume that ε is less than all the mentioned bounds and also $\varepsilon < \varepsilon'(s, \sigma)$. Consider the second family of tori $T_j(2\varepsilon)$ from proposition 3. The $T_j(2\varepsilon)$ are related to complex affine functions f_j and to complex lines $\ell_j = \{f_j = 0\}$.

Labeling the collection of all tori above, i.e. the tori obtained for each k by proposition 2 and the tori $T_j(2\varepsilon)$ from proposition 3, we obtain the family $T_n(2\varepsilon)$. The $T_n(2\varepsilon)$ are pairwise disjoint. Indeed by the choice of δ the tori obtained by proposition 2 for different k do not intersect. Since $\ell_j \cap \partial\mathbb{B}^2 \subset \{|z_1 - (1 - s)| = \frac{5}{3}\sigma\}$ and $\{g_k = 0\} \cap \partial\mathbb{B}^2 \subset \{|z_1 - (1 - s)| \leq \frac{4}{3}\sigma\}$ the tori obtained by proposition 2 do not meet the $T_j(2\varepsilon)$ from proposition 3, if ε is small. The inclusion $(18_{r'})$ from proposition 3 together with (21_r), the latter applied for each k with $r \in [\tilde{r}_k, 1]$, imply (20_r) with r larger than the maximum of all $\tilde{r}_k$ and also larger than r' from proposition 3. Proposition 4 is proved. $\square$

Proposition 4 enables us to construct the sets E_N inductively.

Proof of the theorem.
Step 1. Lemma 1 and proposition 1 give us for any sufficiently small $\varepsilon > 0$ a finite collection of tori. More precisely, we obtain a number r_1 (see remark 2) and a finite collection of numbers $s_m^{(1)}$ such that for each small $\varepsilon > 0$ for each m there exists a finite number of tori unitarily equivalent to $T^{s_m^{(1)}}(\varepsilon)$ with the following properties. Denote all the tori by $T^{(1)}$ skipping labeling indices but indicating that they are obtained at step 1. The $T^{(1)}(2\varepsilon)$ are pairwise disjoint and

$$(22) \qquad \qquad \overline{\beta\mathbb{B}^2} \subset \left(\widehat{\bigcup {}^{r_1} T^{(1)}(\varepsilon)}\right).$$

The number ε will be chosen at the second step of the induction. (Note that the number of tori and the choice of the complex lines which are the symmetry axes of the tori also depend on ε.)

Step 2. For any $s_m^{(1)}$ of step 1 we apply proposition 4. We obtain a bound $\sigma'(s_m^{(1)})$. Choose now the number ε of step 1 (and hence the tori of that step). It has to satisfy the following requirements. First it has to be so small that it fits for step 1. Next for each m we require $\varepsilon < \sigma'(s_m^{(1)})$. Further, for chosen $s = s_m^{(1)}$ and $\sigma = \varepsilon$ proposition 4 allows to find new s-parameters which tend to zero for $\sigma = \varepsilon \to 0$. We denote the collection (over all m) of all new s-parameters at the second step by $s^{(2)}$ (we skip indices labeling them) and require $\sigma = \varepsilon$ being so small that each $s^{(2)}$ is less than $\frac{1}{2^2}$. Finally proposition 4 asserts for each m and ε the existence of a number $\rho(s_m^{(1)}, \varepsilon)$ (see (20)). We wish the estimate

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$1 - \rho(s_m^{(1)}, \varepsilon) > r_1$ for each m. Choose an $\varepsilon > 0$ satisfying all these conditions and denote it by ε_1.

Denote $E_1 = \bigcup T^{(1)}(2\varepsilon_1)$. E_1 is the union of pairwise disjoint closed tori of diameter determined by the $s^{(1)}$ and ε_1. (22) implies that

$$\overline{\beta\mathbb{B}^2} \subset \hat{E}_1.$$

Indeed, $T^{(1)}(2\varepsilon_1) \supset T^{(1)}(\varepsilon_1)$ and $\widehat{T^{(1)}(\varepsilon_1)} \supset {}^{r_1}T^{(1)}(\varepsilon_1)$ for any of the tori $T^{(1)}(\varepsilon_1)$.

The set E_1 is constructed. Describe now the construction of the set E_2 modulo the choice of the parameter ε_2.

Denote the functions corresponding to the $T^{(1)}(\varepsilon_1)$ by $f^{(1)}$ (omitting indices as above). From applying proposition 4 to each of the $T^{(1)}(\varepsilon_1)$ we got the collection of mentioned above numbers $s^{(2)} < \frac{1}{2^2}$ and we obtain a number r_2 (see (20_r)) such that the following holds: For each sufficiently small $\varepsilon > 0$ there exist finitely many pairwise disjoint tori $T^{(2)}(2\varepsilon)$ contained in E_1, each of them unitarily equivalent to a torus $T^{s_m^{(2)}}(2\varepsilon)$ for some of the numbers $s_m^{(2)}$ of the collection $s^{(2)}$, such that (since $1 - \rho(s_m^{(1)}, \varepsilon) > r_1$ for each m)

$$\overline{r_1\mathbb{B}^2} \cap \bigcup\{|f^{(1)}| \leq \varepsilon_1\} \subset \left(\bigcup \widehat{{}^{r_2}T^{(2)}(\varepsilon)}\right),$$

hence by (22)

$$\overline{\beta\mathbb{B}^2} \subset \left(\bigcup \widehat{{}^{r_2}T^{(2)}(\varepsilon)}\right).$$

This describes the construction of the set E_2 modulo the choice of the parameter ε_2.

Step N. This is the general step of the induction. Let $N > 2$. Suppose the sets $E_1 \supset \ldots \supset E_{N-2}$ are constructed with $\hat{E}_{N-2} \supset \overline{\beta\mathbb{B}^2}$ and the construction of the set E_{N-1} is described modulo the choice of the parameter ε_{N-1}. We want to choose the parameter ε_{N-1} and describe the construction of E_N modulo the choice of the ε_N.

More precisely, we suppose the following has been done at step $N - 1$. There was found a finite collection of numbers $s^{(N-1)}$, all less than $\frac{1}{2^{N-1}}$, and a number $r_{N-1} \in (0, 1)$ such that for each sufficiently small $\varepsilon > 0$ there exist finitely many pairwise disjoint tori $T^{(N-1)}(2\varepsilon)$ contained in E_{N-2}, each of them unitarily equivalent to a torus $T^{s_m^{(N-1)}}(2\varepsilon)$ for some number $s_m^{(N-1)}$ of the collection $s^{(N-1)}$, with the following relation for the polynomial hull

(23) $$\overline{\beta\mathbb{B}^2} \subset \left(\bigcup \widehat{{}^{r_{N-1}}T^{(N-1)}(\varepsilon)}\right).$$

We want to choose a suitable number ε_{N-1} for ε. Here are the requirements for the number ε. First it has to be small enough for step $N - 1$ to go through. Further, apply proposition 4 with s to be any of the numbers $s^{(N-1)}$. We obtain an upper bound for σ. Require that $\varepsilon < \sigma'$ the least of all such upper bounds over numbers $s^{(N-1)}$. Now for s equal to any of the $s^{(N-1)}$ and $\sigma = \varepsilon$ proposition 4 allows to choose new s-parameters which tend to zero together with $\sigma = \varepsilon$. Require ε being so small that all the new s-parameters denoted by $s^{(N)}$ are less than 2^{-N}. Finally, the proposition asserts the existence of a

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number $\rho(s_m^{(N-1)}, \varepsilon)$ for each number $s_m^{(N-1)}$ of the $s^{(N-1)}$ (see (20)). We wish the estimate $1 - \rho(s_m^{(N-1)}, \varepsilon) > r_{N-1}$ for each m. Choose a number $\varepsilon > 0$ satisfying all these conditions and denote it by ε_{N-1}. Put

$$E_{N-1} = \bigcup T^{(N-1)}(2\varepsilon_{N-1}).$$

Thus the set E_{N-1} is constructed (recall that also the number of the tori and the choice of their symmetry axes depend on ε_{N-1}) and by (23)

$$\overline{\beta \mathbb{B}^2} \subset \hat{E}_{N-1}.$$

Use now proposition 4 to define E_N modulo the choice of ε_N. Denote the functions corresponding to the tori $T^{(N-1)}(\varepsilon_{N-1})$ of the previous generation by $f^{(N-1)}$. We already mentioned the numbers $s^{(N)}$ obtained by proposition 4. Further this proposition (applied with s being any of the $s^{(N-1)}$ and with $\sigma = \varepsilon_{N-1}$) gives for each sufficiently small $\varepsilon > 0$ finitely many pairwise disjoint tori $T^{(N)}(2\varepsilon)$ contained in E_{N-1} (each of them unitarily equivalent to $T^{s_m^N}(2\varepsilon)$ with s_m^N being some of the numbers s^N) and a number $r_N \in (0,1)$ with the following relation for the polynomial hull

(24) $$\overline{r_{N-1}\mathbb{B}^2} \cap \bigcup \{|f^{(N-1)}| \le \varepsilon_{N-1}\} \subset \left(\bigcup \widehat{r_N T^{(N)}}(\varepsilon) \right).$$

Here we used that $1 - \rho(s_m^{(N-1)}, \varepsilon) > r_{N-1}$ for each m. Hence by (23) with the ε in (23) replaced by ε_{N-1} we have

$$\overline{\beta \mathbb{B}^2} \subset \left(\bigcup \widehat{r_N T^{(N)}}(\varepsilon) \right)$$

for sufficiently small positive ε of step N and the tori $T^{(N)}(\varepsilon)$ the existence of which is obtained at step N.

The induction is complete, hence the theorem is proved. $\qquad\square$

After the paper was written L.Stout informed us that he constructed a Cantor set in $\mathbb{C}^n$ with non-trivial polynomial hull in case $n \ge 4$. He uses purely topological results and the existence of suitable plurisubharmonic Morse functions. His method does not work in dimensions 2 and 3.

Remark 5. *Not every wild Cantor set in the sphere has non-trivial polynomial hull.*

Indeed, this can be seen even by varying the above construction. Let the general step N of the present construction look as follows. Suppose we obtained sets $E_1^* \supset \ldots \supset E_{N-1}^*$, so that the E_k^* are disjoint unions of closed solid tori of the same kind as above of width $2\varepsilon_k^*$. Suppose the polynomial hull of E_k^* is contained in the δ_k-neighbourhood of $(\cup \ell^{(k)}) \cap \overline{\mathbb{B}^2}$ for some small enough positive number δ_k. Here $\ell^{(k)}$ denotes the collection of complex lines which are the symmetry axes of the tori in E_k^*. Choose at step N disjoint tori

$T^{*(N)} \subset E^*_{N-1}$ by doing first the construction of the proof of proposition 4, then fixing the symmetry axes of the tori and taking ε^*_N so small that the polynomial hull of $E^*_N = \cup T^{*(N)}(2\varepsilon^*_N)$ is contained in the δ_N-neighbourhood of $(\cup \ell^{(N)}) \cap \overline{\mathbb{B}^2}$. This is possible since $(\cup \ell^{(k)}) \cap \overline{\mathbb{B}^2}$ is the polynomial hull of $(\cup \ell^{(k)}) \cap \partial \mathbb{B}^2$. If $\delta_N \to 0$ for $N \to \infty$ fast enough then (since $(\cup \ell^{(N)}) \cap \overline{\mathbb{B}^2}$ is close to the sphere for large N) the accumulation points of the mentioned δ_N-neighbourhoods are contained in the sphere. Hence the set $E^* = \cap E^*_N$ is polynomially convex. Using Bing's theorem one can prove that this set is a wild Cantor set.

Remark 6. *The set E constructed in the theorem is rationally convex.*

In fact, each pair of points of E can be separated by a 2-torus T in $\partial \mathbb{B}^2 \setminus E$. Indeed for the 2-torus T we can take the boundary of certain solid torus $T^{(N)}_m(2\varepsilon)$, where $T^{(N)}_m(2\varepsilon_N)$ is one of the tori contributing to E_N for some N and ε is slightly bigger than ε_N. The 2-torus T is contained in the cylinder $\{z \in \mathbb{C}^2 : |f^{(N)}_m(z)| = 2\varepsilon\}$ which is the union of complex lines and does not meet E. Hence, the cylinder does not meet the rational hull of E. Now the same argument works as for polynomial convexity of tame Cantor sets in the sphere.

We do not know whether Cantor sets in the sphere are always rationally convex.

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